On the Polynomial of a Path

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ABSTRACT

Let $A(P_n)$ be the adjacency matrix of the path on n vertices. Suppose that $r(\lambda)$ is a polynomial of degree less than n, and consider the matrix $M = r(A(P_n))$. We determine all polynomials for which M is the adjacency matrix of a graph.

1. INTRODUCTION AND PRELIMINARIES

All graphs will be finite, undirected and without loops or multiple edges. For a graph G, the adjacency matrix, $A(G) = (a_{ij})$, is the 0-1 matrix where $a_{ij} = 1$ if and only if v_i and v_j are adjacent in G. The ith distance graph of G, G_i, has the same vertex set as G, and two vertices are adjacent in G_i if and only if they are a distance i apart in G. For several interesting classes of graphs, $A(G_i)$ is a polynomial in $A(G)$. The best known of these are distance-regular graphs [2, p. 132]. In this case, $A(G_i)$ is a polynomial in $A(G)$ of degree i. Moreover, Weichsel [4] has shown that this property characterizes distance-regular graphs.

Thus, it seems appropriate to ask when a polynomial in the adjacency matrix of a graph will yield the adjacency matrix of another graph. In this note we will investigate this question for a class of trees.

If G and H are graphs, and $r(\lambda)$ is a polynomial such that $r(A(G)) = A(H)$, we shall say that H is generated from G by $r(\lambda)$. The adjacency algebra of G, $\mathscr{A}(G)$, is the algebra of all polynomials in $A(G)$. Define the path on n vertices by $\mathrm{VP}_n = \{v_i \mid 1 \leqslant i \leqslant n\}$ and $\mathrm{EP}_n = \{[v_i, v_{i+1}] \mid 1 \leqslant i \leqslant n - 1\}$. Let $\chi(P_n; \lambda)$ denote the characteristic polynomial of P_n, and define $\chi(P_0; \lambda) = 1$. We will write the entry in position (i, j) of a matrix M as $m_{i,j}$.

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2. THE MATRICES D_m

Throughout this section we shall consider that n is given, and we define D_m for $0 \leqslant m \leqslant n-1$ by $D_m = \chi(P_m; A(P_n))$.

LEMMA 2.1. *For $0 \leqslant m \leqslant n-1$, D_m is a 0-1 matrix and $d_{m,i,j} = 1$ if and only if* (a) $i - j \equiv m \pmod 2$, (b) $|i-j| \leqslant m$, *and* (c) $m+2 \leqslant i+j \leqslant 2n-m$.

Proof. We proceed by induction on m. It is easy to see that the result is true for $m=0$ and $m=1$.

The characteristic polynomial of P_m satisfies the recursion $\chi(P_m; \lambda) = \lambda\chi(P_{m-1}; \lambda) - \chi(P_{m-2}; \lambda)$ [3]. Thus we find

$$\begin{aligned} D_m &= A(P_n)D_{m-1} - D_{m-2}, \\ d_{m,i,j} &= d_{m-1,i-1,j} + d_{m-1,i+1,j} - d_{m-2,i,j} \end{aligned} \tag{2.1}$$

unless $i=1$ or $i=n$; in that case the application of the criteria of Lemma 2.1 will give $d_{m-1,0,j} = d_{m-1,n+1,j} = 0$.

Now we assume that the lemma is true for $m = 0,1,2,\ldots,t-1$ and use (2.1) to determine the entries of D_t. There are several cases to consider.

Case 1: $i - j \not\equiv t$ (mod 2). In this case the induction hypothesis gives

$$d_{t-1,i-1,j} = d_{t-1,i+1,j} = d_{t-2,i,j} = 0,$$

so (2.1) shows that $d_{t,i,j} = 0$.

Case 2: $|i-j| > t$, $i-j \equiv t$ (mod 2). When $i-j > t$, $d_{t-1,i-1,j} = d_{t-1,i+1,j} = d_{t-2,i,j} = 0$, so $d_{t,i,j} = 0$. Similarly, when $i-j < -t$, $d_{t,i,j} = 0$.

Case 3: $|i-j| = t$, $t+2 \leqslant i+j \leqslant 2n-t$, $i-j \equiv t$ (mod 2). When $i-j = t$, $d_{t-1,i-1,j} = 1$ and $d_{t-1,i+1,j} = d_{t-2,i,j} = 0$, so $d_{t,i,j} = 1$. When $i-j = -t$, $d_{t,i,j} = 1$.

Case 4: $|i-j| < t$, $t+2 \leqslant i+j \leqslant 2n-t$, $i-j \equiv t$ (mod 2). In this case we find that $d_{t-1,i-1,j} = d_{t-1,i+1,j} = d_{t-2,i,j} = 1$, so $d_{t,i,j} = 1$.

Case 5: $|i-j| < t$, $i+j = t$ or $i+j = 2n-t+2$, $i-j \equiv t$ (mod 2). When $i+j=t$, $d_{t-1,i-1,j} = 0$ and $d_{t-1,i+1,j} = d_{t-2,i,j} = 1$, so $d_{t,i,j} = 0$. Similarly, when $i+j = 2n-t+2$, $d_{t,i,j} = 0$.

Case 6: $i+j < t$ or $i+j > 2n-t+2$, $i-j \equiv m$ (mod 2). When $i+j < t$, $d_{t-1,i-1,j} = d_{t-1,i+1,j} = d_{t-2,i,j} = 0$, so $d_{t,i,j} = 0$. When $i+j > 2n-t+2$, $d_{t,i,j} = 0$.

These six cases prove the validity of the lemma for $m = t$, and thus concludes the induction. ■

LEMMA 2.2. *The set* $\{D_m \mid 0 \leqslant m \leqslant n-1\}$ *is a basis of* $\mathscr{A}(P_n)$.

Proof. Since P_n has diameter $n-1$, the matrices I_n, $A(P_n)$, $A(P_n)^2, \dots, A(P_n)^{n-1}$ are linearly independent [2, p. 12]. Since the dimension of $\mathscr{A}(P_n)$ is at most n, these powers of $A(P_n)$ are a basis of $\mathscr{A}(P_n)$.

Define the linear transformation ψ from $\mathscr{A}(P_n)$ to $\mathscr{A}(P_n)$ by $\psi(A(P_n)^m) = D_m$, $0 \leqslant m \leqslant n-1$. Since $\chi(P_m; \lambda)$ is a polynomial of degree m, $D_m \in \mathscr{A}(P_n)$ and ψ has full rank. Thus, the image of the basis $\{A(P_n)^m \mid 0 \leqslant m \leqslant n-1\}$, $\{D_m \mid 0 \leqslant m \leqslant n-1\}$, is a basis of $\mathscr{A}(P_n)$. ■

3. THE MAIN RESULT

Define the integer l by $l = \lfloor n/2 \rfloor$.

THEOREM 3.1. *Suppose that* $r(\lambda)$ *is a polynomial of degree less than* n. *Then* $r(\lambda)$ *generates a nonnull graph from* P_n *if and only if* $r(\lambda) = \chi(P_{2i+1}; \lambda)$ *for some* i, $0 \leqslant i \leqslant l-1$.

Proof. First, suppose that $r(\lambda) = \chi(P_{2i+1}; \lambda)$ for some i, $0 \leqslant i \leqslant l-1$. Then $r(A(P_n)) = D_{2i+1}$. From Lemma 2.1, it is apparent that D_{2i+1} is a symmetric 0-1 matrix with a zero diagonal, and therefore is the adjacency matrix of a graph.

Now, for the converse, assume that $r(\lambda)$ generates a graph from P_n. For convenience, let $M = r(A(P_n))$. Then M is 0-1 matrix, with a zero diagonal. Since $M \in \mathscr{A}(P_n)$, Lemma 2.2 implies that there are numbers a_j, $0 \leqslant j \leqslant n-1$, such that $M = \sum_{j=0}^{n-1} a_j D_j$.

Assume that $a_{2i} \neq 0$ for some i, $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$. Then let s be the smallest integer such that $a_{2s} \neq 0$. Then,

$$m_{s+1,s+1} = \sum_{j=0}^{n-1} a_j d_{j,s+1,s+1} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{2i} d_{2i,s+1,s+1}$$

$$= \sum_{i=0}^{s} a_{2i} d_{2i,s+1,s+1} = a_{2s} d_{2s,s+1,s+1} = a_{2s}.$$

Since M has a zero diagonal, $0 = m_{s+1,s+1} = a_{2s}$, which contradicts the choice of s. Therefore, $a_{2i} = 0$, $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$.

Now, notice that

$$m_{1,2i+2} = \sum_{k=0}^{l-1} a_{2k+1} d_{2k+1,1,2i+2} = a_{2i+1}, \qquad 0 \leqslant i \leqslant l-1.$$

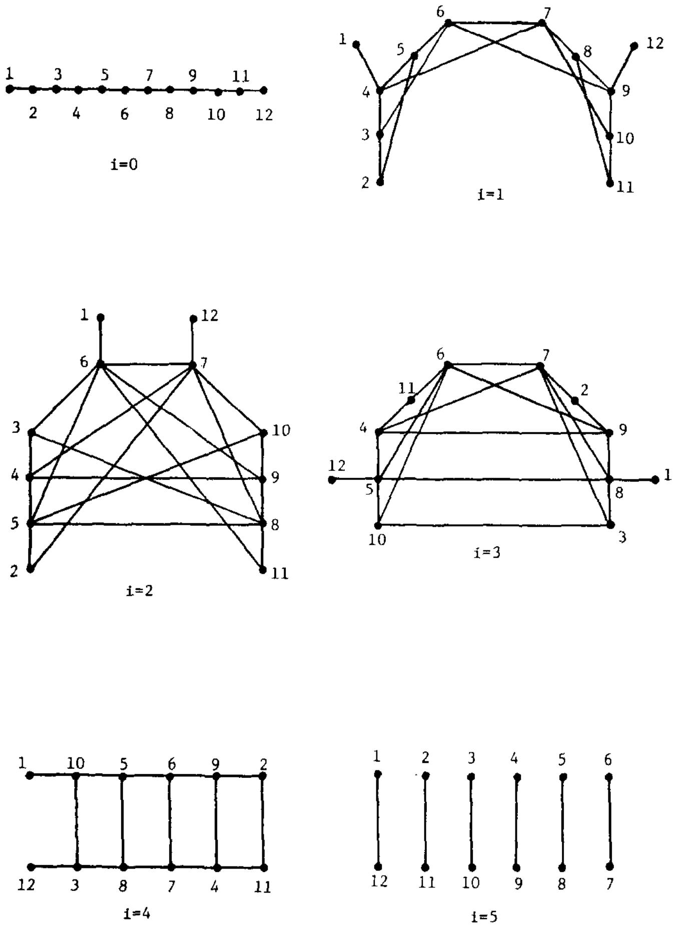

FIG. 1. Graphs with adjacency matrix D_{2i+1}, $0 \leqslant i \leqslant 5$, $n = 12$.

Because M is a 0-1 matrix, $a_{2i+1} \in \{0,1\}$, $0 \leqslant i \leqslant l-1$. Then

$$m_{l+1,l} = \sum_{k=0}^{l-1} a_{2k+1} d_{2k+1,l+1,l} = \sum_{k=0}^{l-1} a_{2k+1}.$$

This relation implies that at most one of a_{2i+1}, $0 \leqslant i \leqslant l-1$, is equal to 1. If they are all zero, then $M = 0$, contrary to the hypothesis. Therefore, $M = D_{2i+1}$ for some i, $0 \leqslant i \leqslant l-1$.

The minimum polynomial of P_n has degree n, $r(\lambda)$ has degree less than n, and $\chi(P_{2i+1}; A(P_n)) = D_{2i+1} = r(A(P_n))$, so $r(\lambda) = \chi(P_{2i+1}; \lambda)$ for some i, $0 \leqslant i \leqslant l-1$. ∎

4. CONCLUSION

The graphs which have D_{2i+1} as their adjacency matrix are displayed in Figure 1 for $n = 12$ and $0 \leqslant i \leqslant 5$.

A computer study was performed to determine if there are any other situations similar to Theorem 3.1. The only case of interest involved sunsets. A sunset is a path of even length greater than two, with additional vertices adjacent to the central vertex. There exists a nontrivial polynomial which generates a graph from a sunset [1]. It is interesting that the polynomial is a divisor of the characteristic polynomial of a subgraph of the sunset, and the generated graph is isomorphic with the original sunset.

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REFERENCES

1 R. A. Beezer, Polynomials of the adjacency matrix of a graph, Ph.D. Thesis, Univ. of Illinois, Urbana, Ill., 1984.
2 N. L. Biggs, *Algebraic Graph Theory*, Cambridge U.P., Cambridge, 1974.
3 D. M. Cvetkovic, M. Doob, and M. Sachs, *Spectra of Graphs*, Academic, New York, 1980, p. 59.
4 P. M. Weichsel, On distance-regularity in graphs, *J. Combin. Theory Ser. B* 32:156–161 (1982).

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